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ON THE PROBLEM OF THE RELATIONSHIP BETWEEN THE SCHWARZSCHILD AND TOLMAN METRICS

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The problem of the relationship between the Schwarzschild and Tolman metrics has occupied the attention of many workers. Although the solutions given in [1 – 3] satisfy the equations of the general relativity theory (OTO) (*), they contradict the correspondence principle. This means that for $G \to 0$, the interval is not transformed into the interval of the special relativity theory (CTO) (*), while for $c \to \infty$, the solutions do not become Newtonian. This is apparently caused by the unfortunate choice of the coordinates in the Tolman frame of reference. Papers [4, 5] illustrate particular cases of a correct passage from one metric to the other.

In the present paper a general method of obtaining solutions is proposed in which the passage from one frame of reference to the other satisfies the correspondence principle.

The intervals in the co-moving frame of reference and in the central frame of reference are, respectively,

$$-ds^2 = -c^2d\tau^2 + e^{\omega}dR^2 + r^2d\Omega^2 \tag{1}$$

$$-ds^2 = -e^{\nu}c^2dt^2 + e^{\lambda}dr^2 + r^2d\Omega^2 \tag{2}$$

$$(d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2)$$

Since $r = r(c\tau, R)$ and $ct = ct(c\tau, R)$, we have

$$dr = r^{\cdot}cd\tau + r'dR, \quad cdt = ct^{\cdot}cd\tau + ct'dR$$

$$(r^{\cdot} = \partial r / c\partial\tau, \ r' = \partial r / \partial R, \ ct^{\cdot} = c\partial t / c\partial\tau, \ ct' = c\partial t / \partial R)$$

Substituting these differentials into (2), equating the coefficients accompanying $c^2d\tau^2$ and dR^2 and remembering that the coefficient of $2cd\tau dR$ is zero, we obtain

$$e^{\nu}c^2t^{\cdot 2} - r^{\cdot 2}e^{\lambda} = 1, \qquad e^{\lambda}r'^2 - c^2t'^2e^{\nu} = e^{\omega}, \qquad e^{\lambda}r^{\cdot}r' - ct^{\cdot}ct'e^{\nu} = 0 \tag{3}$$

from which, eliminating e^{λ} and e^{ν}, we have

$$e^{\lambda} = e^{\omega} / (r'^2 - r^{\cdot 2}e^{\omega}), \quad e^{\nu} = r'^2 / [c^2t^{\cdot 2}(r'^2 - r^{\cdot 2}e^{\omega})] \tag{4}$$

$$(e^{\omega}ct^{\cdot}r^{\cdot} - ct'r')(ct^{\cdot}r' - ct'r^{\cdot}) = 0$$

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*) Editors note. The abbreviations (OTO) and (CTO) are used in the relevant Soviet literature and stand for "general relativity theory" and "special relativity theory", respectively.

The expression

$$\Delta_0 = r' ct^{\cdot} - r^{\cdot} ct' = \frac{\partial (r, ct)}{\partial (R, c\tau)} = \exp \frac{\omega - (\lambda + \nu)}{2} \tag{5}$$

is the Jacobian of the transformation from the system (1) to the system (2). The transformation is allowed when $\Delta_0 \neq 0$, therefore from (4) we have

$$e^{\omega} ct^{\cdot} r^{\cdot} - ct' r' = 0 \tag{6}$$

In the computations that follow, it is convenient to pass to the independent variables R and r. Then

$$r^{\cdot} = \frac{1}{c\tau_r}, \qquad r' = -\frac{c\tau_R}{c\tau_r}, \qquad ct^{\cdot} = \frac{ct_r}{c\tau_r}, \qquad ct' = ct_R - \frac{c\tau_R}{c\tau_r} ct_r$$

In the Tolman solution $(p = 0)$ with the metric (1) we have [2]

$$c\tau_r = \frac{1}{r^{\cdot}} = \left[f(R) + \frac{F(R)}{r} \right]^{-1/2}, \qquad c\tau - c\tau_0(R) = \int \left(f + \frac{F}{r} \right)^{-1/2} dr$$

Moreover, we can easily compute $c\tau_R$. In fact,

$$c\tau_{Rr} = -\frac{1}{2} \left(f' + \frac{F'}{r} \right) \left(f + \frac{F}{r} \right)^{-3/2}$$

$$c\tau_R - c\tau_{0R} = -\frac{1}{2} \int \left[\left(f' + \frac{F'}{r} \right) \left(f + \frac{F}{r} \right)^{-3/2} \right] dr$$

We have the following relations

$$\int \left(f + \frac{F}{r} \right)^{-1/2} dr = \left(f + \frac{F}{r} \right)^{-1/2} r - \frac{F}{2} \int \left(f + \frac{F}{r} \right)^{-3/2} \frac{dr}{r}$$

$$\int \left(f + \frac{F}{r} \right)^{-3/2} \frac{dr}{r} = \frac{2}{F} \left[\left(f + \frac{F}{2} \right)^{-1/2} r - (c\tau - c\tau_0) \right]$$

$$\int \left(f + \frac{F}{r} \right)^{-3/2} f dr = \int \left(f + \frac{F}{r} \right)^{-1/2} dr - F \int \left(f + \frac{F}{r} \right)^{-3/2} \frac{dr}{r}$$

Consequently

$$\int \left(f + \frac{F}{r} \right)^{-3/2} dr = \frac{1}{f} \left\{ (c\tau - c\tau_0) - 2 \left[\left(f + \frac{F}{2} \right)^{-1/2} r - (c\tau - c\tau_0) \right] \right\} =$$
$$\frac{1}{f} \left[3(c\tau - c\tau_0) - \left(f + \frac{F}{2} \right)^{-1/2} 2r \right]$$

and therefore

$$c\tau_R - c\tau_{0R} = \frac{f'}{f} \left[-\frac{3}{2} (c\tau - c\tau_0) + \left(f + \frac{F}{r} \right)^{-1/2} r \right] +$$
$$\frac{F'}{F} \left[-\left(f + \frac{F}{r} \right)^{-1/2} r + (c\tau - c\tau_0) \right]$$

For $F = r_g = \text{const}$, we have

$$c\tau_R - c\tau_{0R} = \frac{f'}{f} \left[-\frac{3}{2} (c\tau - c\tau_0) + \left(f + \frac{r_g}{r} \right)^{-1/2} r \right] \tag{7}$$

From (6) we obtain

$$e^{\omega} = (c\dot{\tau}_R ct_r - ct_R c\tau_r) \frac{c\tau_R}{ct_r} = \frac{c\tau_R}{ct_r} \frac{\partial (c\tau, ct)}{\partial (R, r)}$$

and

$$\Delta_0 = ct_R / ct_r = -r' (ct_R / c\tau_R)$$

Further from (3) we find

$$ct_r = (c^2\tau_r^{\,2} + e^{\lambda})^{1/2} e^{-1/2\nu}, \; ct_R / c\tau_R = c\tau_r \, [e^{\nu}(e^{\lambda} + c^2\tau_r^{\,2})]^{-1/2}$$

and this finally gives

$$e^{\omega} = c^2\tau_R^{\,2}(c^2\tau_r^{\,2})^{-1}\left(e^{-\lambda} + \frac{1}{c^2\tau_r^{\,2}}\right)^{-1} = c^2\tau_R^{\,2} e^{\lambda}(c^2\tau_r^{\,2} + e^{\lambda})^{-1}$$

For $e^{\nu} = e^{-\lambda} = 1 - r_g/r$ and $F = r_g$, we have

$$ct_r = (1+f)^{1/2}\left(1 + \frac{r_g}{r}\right)^{-1/2}\left(1 - \frac{r_g}{r}\right)^{-1}, \; ct_R = c\tau_R(1+f)^{-1/2}$$

$$e^{\omega} = r'^2(1+f)^{-1} = \Delta_0^{\,2}$$

The converse problem is also easily solved. Knowing

$$r_{c\tau} = \left(f + \frac{F}{r}\right)^{1/2}, \qquad e^{\omega} = r'^2(1+f)^{-1}$$

we find from (4)

$$e^{\lambda} = \left(1 - \frac{F}{r}\right)^{-1}, \; e^{\nu} = (1+f)\left[c^2t^{\cdot 2}\left(1 - \frac{F}{r}\right)\right]^{-1} =$$

$$(1+f)\left(1 - \frac{F}{r}\right)^{-1}\left[c^2t_{\tau}^{\,2}\left(f + \frac{F}{r}\right)\right]^{-1}$$

From this it follows that

$$ct^{\cdot} = (1+f)^{1/2}\exp\frac{\lambda - \nu}{2}$$

or

$$ct_r = (1+f')^{1/2}\left(\exp\frac{\lambda-\nu}{2}\right)c\tau_r$$

and

$$ct_r\left(f + \frac{F}{r}\right)^{1/2}\left(1 - \frac{F}{r}\right)(1+f)^{-1} = \frac{ct_R}{c\tau_R} \tag{8}$$

$$\frac{u}{c} = r_{ct}\exp\frac{\lambda-\nu}{2} = r_{c\tau}(1+f')^{-1/2} = \left(f + \frac{F}{r}\right)^{1/2}(1+f)^{-1/2}$$

This determines

$$ct = ct(r, R), \; e^{\nu} = \left(1 - \frac{F}{r}\right)c^2\tau_R^{\,2}[(1+f)c^2t_R^{\,2}]^{-1}$$

$$ct^{\cdot} = \frac{ct_R}{c\tau_R}(1+f)\left(f + \frac{F}{r}\right)^{-1}$$

For $F = r_g$, we have

$$e^{\lambda} = \left(1 - \frac{r_g}{r}\right), \quad e^{\nu} = 1 - \frac{r_g}{r}, \quad c\tau_R = (1+f)^{1/2}ct_R,$$

$$ct^{\cdot} = (1+f)^{1/2}\left(1 + \frac{r_g}{r}\right)^{-1}$$

When solving the Friedman problem we have, e.g. for a closed model for $p = 0$

$$e^{\omega} = a^2/4a_0^{\,2}, \; r = a\sin X, \; R = 2a_0X, \; F = 2a_0\sin^3 X$$

$$e^{\lambda} = \left(1 - \frac{2a_0}{a}\sin^2 X\right)^{-1}$$

and Eqs. (8) determines $ct = ct(r, R)$, after which e^{ν} is determined.

Let us now solve the Schwarzschild problem in the Tolman metric, using the following important principles.

1) Since at the limits the Einstein equations pass into the interval of the special relativity theory and of Newtonian laws, and then to that of Galilean mechanics, and because the sequence of such transitions is immaterial, these requirements (the principle of correspondence) must be also satisfied in the Tolman metric.

2) The Jacobian $\Delta_0 \neq 0$ of the transformation does not tend to infinity over the whole region in question (except at the singularities), nor during the limiting passages to the special relativity theory, to Newton's theory and to the Galilean transformations.

3) If the given region is covered in one frame of reference by n "maps", then this number is preserved in the other frame of reference. Since only the allowed coordinate systems are used here, the class of functions remains unchanged on the passage from one system to the other. It is usually assumed that the coordinate transformation X^i refers to the class of functions $c^{(2)}$ and the transformations g_{ik} to the class $c^{(1)}$.

Since $c\tau_r = (f + r_g / r)^{-1/2}$, where $1 + f > 0$, then

$$\begin{aligned} f\,(c\tau - c\tau_0) &= (fr^2 + r_g r)^{1/2} - r_g\,(-f)^{-1/2} \arcsin\,(-rf / rq)^{1/2}, \quad f < 0 \\ {}^3/_2 r_g{}^{1/2}\,(c\tau - c\tau_0) &= r^{3/2}, \quad f = 0 \\ f\,(c\tau - c\tau_0) &= (fr^2 + r_g r)^{1/2} - r_g\,(f)^{-1/2} \operatorname{arcsh}\,(r_f / r_g)^{1/2}, \quad f > 0 \end{aligned} \tag{9}$$

Let us consider these three modes of motion separately.

1°. For $f < 0$, we have an "elliptic" motion. For $c\tau = 0$, it is necessary and sufficient to set $r = R$ (the Euler and Lagrange coordinates coincide) and

$$dr / cd\tau = (v / c) = (f + r_g / r)^{1/2} = 0$$

and this yields all possible motions. From the above conditions we have

$$f = -\frac{r_g}{R}, \quad c\tau_0 = -\frac{R^{3/2}}{r_g^{1/2}}\frac{\pi}{2}, \quad v = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{R}\right)} \tag{10}$$

$$\sqrt{r_g}\, c\tau = \sqrt{2CM}\,\tau = R^{3/2}\left[\arcsin\sqrt{\frac{r}{R}} - \frac{\pi}{2} - \sqrt{\frac{r}{R}\left(1 - \frac{r}{R}\right)}\right]$$

We note that R is determined with the accuracy up to a consonant, therefore we can e.g. set $r = R + r_g$ for $c\tau = 0$. Then in the formulas (10) R should be replaced by $R + r_g$. In the present case $\lambda + \nu = 0$, therefore it follows from (5)

$$\Delta_0 = \exp\frac{\omega}{2} = \frac{r'}{\sqrt{1+f}} = -c\tau_R\left(\frac{1 + r_g / r}{1 + f}\right)^{1/2} \tag{11}$$

where $c\tau_R$ is determined by (7).

Since

$$f = -r_g\,(R + r_g)^{-1}, \quad c\tau_0 = -(R + r_g)^{3/2} r_g^{-1/2}\frac{\pi}{2}$$

then

$$\frac{f'}{f} = -(R + r_g)^{-1}, \qquad c\tau_{0R} = -\frac{3\pi}{4}(R + r_g)^{1/2} r_g^{-1/2} \tag{12}$$

$$\Delta_0 = \frac{3}{2}\sqrt{\frac{R + r_g - r}{rR}}\left[\sqrt{R + r_g}\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{r}{R + r_g}}\right) + \right.$$

$$\sqrt{r\left(1-\frac{r}{R+r_g}\right)}\Bigg]+r\,[R\,(R+r_g)]^{-1/2}$$

It can easily be verified that (12) determines the Jacobian Δ_0 which satisfies all the requirements formulated above. Thus the solution given is perfectly correct. The Jacobian Δ_0 is finite everywhere except at the center $R = 0$ which naturally is a singularity for $r_g = 0$ ($G = 0$ or $c \to \infty$), $R = r$, and $\Delta_0 = 1$.

We now turn our attention to the solution of the present problem given by Novikov in [1]. In his solution the scale of the Lagrangian coordinate chosen is different, namely $f = -r_g^2(\overline{R}^2+\tau_g^2)^{-1}$. His Lagrangian coordinate $\overline{R}$ is related to the coordinate used in the present paper as follows: $\overline{R}^2 = Rr_g$, $dR\,/\,d\overline{R} = 2\overline{R}\,/\,r_g$ and we also have

$$c\tau_0 = -\frac{\pi}{2}\frac{(\overline{R}^2+r_g^2)^{3/2}}{r_g^2}, \qquad \frac{f'}{f} = -\frac{2\overline{R}}{\overline{R}^2+r_g^2}$$

$$c\tau_0\overline{R} = -\frac{3\pi}{2}(\overline{R}^2+r_g^2)^{1/2}\frac{\overline{R}}{r_g^2}$$

By choosing the Lagrangian coordinate in this manner, the author of [1] succeeded in achieving the so-called completeness for his solution. As the result of this completeness, the world line of any particle moving in the Schwarzschild's field either lies on a central singularity ($r = 0$), or vanishes at infinity. Moreover, the solution of maximum completeness is an analog of the Kruskal solution and this, in the opinion of the author, is a significant achievement. It can however be shown that this completeness contradicts the correspondence principle. The Novikov coordinate system is inadmissible in this sense for the reason of ambiguity in the choice of the Lagrangian's coordinate: for $r_g \to 0$ ($G = 0$ or $c \to \infty$), $\Delta_0 \to \infty$. In fact, Novikov gives

$$\Delta_0 = \frac{3}{2\overline{R}}\sqrt{\frac{\overline{R}^2+r_g^2-rr_g}{r}}\left[\sqrt{\frac{\overline{R}^2+r_g^2}{r_g}}\times\right.$$

$$\left.\left(\frac{\pi}{2}-\arcsin\sqrt{\frac{rr_g}{\overline{R}^2+r_g^2}}\right)+\sqrt{r\left(1-\frac{rr_g}{\overline{R}^2+r_g^2}\right)}\right]+\frac{rr_g}{\overline{R}\sqrt{\overline{R}^2+r_g^2}}$$

and when $r_g \to 0$, we have

$$\Delta_0 = \frac{3\pi}{4}\frac{\overline{R}}{\sqrt{rr_g}} \to \infty, \qquad \overline{\Delta} = \frac{\partial\,(r, ct)}{\partial\,(\overline{R}, c\tau)} = \Delta_0\frac{dR}{d\overline{R}} = \frac{3\pi}{4}\frac{\overline{R}}{\sqrt{rr_g}}\times$$

$$\frac{2\overline{R}}{r_g} = \frac{3}{2}\pi\frac{\overline{R}^2}{r^{1/2}r_g^{3/2}} \to \infty$$

In [2] the authors assert that the coordinates (r, t, or any others) can be subjected to any transformation. It is this assertion that was used by Novikov. It is however known that the coordinate transformations cannot be arbitrary; they must preserve the calss of functions. Novikov obtains two spaces: one for $\overline{R} > 0$ and another for $\overline{R} < 0$, and the second space adjoins the first one along the line $\overline{R} = 0$. But in this case $dR/d\overline{R} \to 0$, the uniqueness is clearly violated and the "second" space is merely a mathematical duplicate of the "first" one. In our formalism, there is no "second" space.

Further, when $c \to \infty$ or $G = 0$, the correspondence principle is not satisfied in the Novikov's work and $v^2 = 2GM\,[(1\,/\,r) - r_g\,/\,(\overline{R}^2 + \tau^2)]$. For $c \to \infty$, $v^2 = 2GM\,/\,r$ and, in contrast to our formalism, all elliptic motions vanish on passing to the Schwartzschild theory. It would seem that the Novikov's requirements concerning the physical sense of two (or four) spaces in a Kruskal type metric can be reconciled with the observance of the correspondence principle if we set $f = -r_g (R^{*2} + r_g)^{-1/2}$. Then

$$R = -r_g \pm (R^{*2} + r_g^2)^{1/2}$$

$$c\tau_0 = -\frac{\pi}{2} r_g^{-1/2} (R^{*2} + r_g^2)^{3/4}, \qquad v^2 = 2GM \, [r^{-1} - (R^{*2} + r_g^2)^{-1/2}]$$

and all correspondence principles hold. But

$$\frac{f_{R^*}}{f} = -\frac{R^*}{(R^{*2} + r_g^2)^{1/2}}, \qquad c\tau_{0R^*} = -\frac{3\pi}{4} \frac{R^*}{r_g^{1/2} (R^{*2} + r_g^2)^{3/4}}$$

Therefore when $R^* = 0$

$$\Delta_0^* = \frac{\partial (r, ct)}{\partial (R^*, c\tau)} = \Delta_0 \frac{dR}{dR^*} = \Delta_0 \frac{R^*}{(R^* + r_g^2)^{1/2}} = 0$$

and the "second" space is again found to be a mathematical duplicate of the "first" one.

2°. For $f = 0$ we assume that when $c\tau = 0$, $r = R + r_g$ (or $r = R$, it makes no difference), then we have

$$r^{3/2} = (R + r_g)^{3/2} \pm {}^3/_2 r_g^{1/2} c\tau, \qquad c\tau_0 = \frac{2}{3} \frac{(R + r_g)^{3/2}}{r_g^{-1/2}}$$

$$v^2 = 2GM / r, \qquad r' = (R + r_g)^{1/2} r^{-1/2} = \Delta_0$$

This solution fulfills all the requirements given above.

Earlier, Lemaitre and Rylov [3] made the substitution $c\tau_0 = -R$ which gave $r^{3/2} = {}^3/_2 (r_g)^{1/2} (R \pm c\tau)$, but in this case $r' = (r_g r^{-1})^{1/2} = \Delta_0$. Assuming $c \to \infty$ or $G = 0$ we see that in this case the correspondence principle also is not fulfilled, consequently the choice of $c\tau_0$ is unsatisfactory.

3°. For $f > 0$ setting $c\tau = 0$ and $r = R + r_g$ (or $r = R$), we specify that when $r \to \infty$, $v/c = f^{1/2} = v_0 (R)' / c$ which defines $f = v_0^2 / c^2$. Further, setting $c\tau = 0$ we find from (9) $c\tau_0$ and this solves the present problem completely

$$v_0 \tau = \left[r \left(r \frac{v_0}{c} + r_g \right) \right]^{1/2} - \left[(R + r_g) \left[(R + r_g) \frac{v_0}{c} + r_g \right] \right]^{1/2} - \frac{r_g c}{v_0} \operatorname{arcsh} \sqrt{\frac{v_0 r}{c r_g}} + \frac{c r_g}{v_0} \operatorname{arcsh} \sqrt{\frac{v_0}{c r_g} (R + r_g)}$$

Let us now compute the three-velocity in the central frame of reference. We know that

$$\frac{u^2}{c^2} = \frac{f + r_g / r}{1 + f}, \qquad u = c \text{ when } r = r_g$$

$$u^2 = \begin{cases} 2GMr^{-1}, & f = 0 \\ 2GMr^{-1} (R + r_g - r) R^{-1}, & f < 0 \\ (2GMr^{-1} + v_0^2) / (1 + v_0^2 c^{-2}), & f > 0 \end{cases}$$

Since $u = c$ when $r = r_g$, this implies that the Schwartzschild sphere is a real singularity. The energy of any sample particle for the metric (1)

$$E = \frac{\sqrt{-g_{00}} \, E_0}{\sqrt{1 - u^2 c^{-2}}} = \frac{E_0 \sqrt{1 + f}}{\sqrt{1 - r_g / r}}$$

tends to infinity ($\sqrt{-g_{00}} = 1$). In the Schwartzschild metric we had a coordinate singularity but the energy in the constant field was conserved, in the Tolman metric the coordinate singlarity is removed but the energy causes certain difficulties at the Schwartzschild radius.

Since

$$\overline{R}^2 = R_{ilkm}R^{ilkm} = 12r_g^2 / r^6 \simeq \varkappa^2\varepsilon_g^2$$

where ε_g is the density of the gravitational field energy, we have

$$\varepsilon_g = \frac{\sqrt{12}\, r_g}{\varkappa r^3} = \frac{\sqrt{3}\, Mc^8}{2\pi r^2}$$

For $r = r_g$ we have

$$\varepsilon_g = \frac{\sqrt{12}}{\varkappa r_g^2} = \frac{\sqrt{3}\, c^8}{16\pi G^3 M^2}$$

while for $M = m_L = [c\hbar\,(2G)^{-1}]^{1/2} = 10^{-5}$ g, we have

$$\varepsilon_g \simeq c^7 / G^2\hbar \simeq 10^{115} \text{ erg/cm}^3$$

which corresponds to the density of the quantum-gravitational plankton particle.

For $M \simeq 10^{35}$ which corresponds to the mass of the proposed black hole we obtain $\varepsilon_g \simeq 35$ erg/cm^3 which corresponds to the energy density of a nucleon. When the mass is less than 10^{35}, the quantity ε_g exceeds the energy density given above. But at such densities of energy the classical theory of gravitation is not applicable any more and the quantum theory must be used instead.

We can derive a basic conclusion stating that the investigation of singularities on the Schwartzschild sphere cannot be performed within the framework of the classical theories. The general theory of relativity is valid for an external field only when $r > r_g$.

We note that the metric (1) can be reduced by coordinate transformation to another form suitable for analysis. Let $c\tau = c\tau\,(r, R)$, then $cd\tau = c\tau_r dr + c\tau_R dR$. From (1) we have

$$-ds^2 = -2c\tau_r c\tau_R drdR + (e^{\omega} - c^2\tau_R^2)\,dR^2 - c^2\tau_r^2 dr^2 + r^2 d\Omega^2 \qquad (13)$$

hence, we find e.g. for the case $f = 0$, $r^{3/2} = R^{3/2} + {}^3/_2 r_g^{1/2}\, c\tau$, that

$$c\tau_r = (rr_g^{-1})^{1/2},\quad c\tau_R = -(Rr_g^{-1})^{1/2},\quad e^{\omega} = R / r$$

Thus the metric (13) can be reduced to

$$-ds^2 = 2\,(rR)^{1/2} r_g^{-1} drdR + Rr_g^{-1}\,(r_g r^{-1} - 1)\,dR^2 - rr_g^{-1}dr^2 + r^2 d\Omega^2 \qquad (14)$$

Introducing new coordinates

$$dr_1 = (rr_g^{-1})^{1/2} dr,\qquad dR_1 = (Rr_g^{-1})^{1/2} dR$$

we obtain from (14)

$$-ds^2 = \left[\left(\frac{2r_g}{3r_1}\right)^{2/3} - 1\right] dR_1^2 + 2dr_1\,dR_1 - dr_1^2 + r_g^2\left(\frac{2r_g}{3r_1}\right)^{-4/3} d\Omega^2$$

Further, setting

$$(2rg)^2\,(3r_1)^{-2} = r^2{}_g r^{-3},\quad dR_1 = dR$$

we can finally write the metric in question, using the coordinates r and R, in the form

$$-ds^2 = -\left(1 - \frac{r_g}{r}\right) dR^2 + 2\,\frac{r^{1/2}}{r_g^{1/2}}\, dr\,dR - \frac{r}{r_g^1}\, dr^2 + r^2 d\Omega^2$$

The singularity appearing in this metric at $r = r_g$ is retained in the coefficient accompanying dR^2.

Let us see what results can be obtained from a closed, isotropic Friedman's model. Since $r = a\sin\chi$ and $R = 2a_0\chi$, we easily obtain

$$c\tau_r = \frac{1}{a^{\cdot}\sin\chi}, \qquad c\tau_R = -\frac{r\cos\chi}{2a^{\cdot}a_0\sin^2\chi}$$

Knowing that $r_\chi = a\cos\chi$, we have $r' = r_R = a\,(2a_0)^{-1}\cos\chi$. At the same time from $e^\omega = r'^2 / (1 + f)$ (remembering that $f = -\sin^2\chi$), we find $e^\omega = (a / 2a_0)^2$. Substituting the values for $c\tau_r$, $c\tau_R$ and e^ω into (13) we have

$$-ds^2 = \frac{r\cos\chi}{a^{\cdot 2}a_0\sin^2\chi}\,dr\,dR + \frac{a^2}{4a_0^2}\left(1 - \frac{\cos^2\chi}{a^{\cdot 2}\sin^2\chi}\right)dR^2 - \frac{dr^2}{a^{\cdot 2}\sin^2\chi} + r^2 d\Omega^2$$

which is convenient e.g. for writing out the equations $T^k_{i,k} = 0$ when two quasi-linear equations defining ε and u can be obtained simultaneously.

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ON REGULAR PRECESSIONS OF A HEAVY GYROSTAT

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The regular precessions of a heavy asymmetric gyrostat are found by direct integration of the system of Zhukovskii equations written in the principal axes of inertia. The properties of these motions are investigated; the possibility of controlling them is revealed. Forces capable of causing a regular precession in the gyrostats are investigated.

An idea was developed in [1] on the preferability of investigating the motion of a heavy gyrostat fixed at one point before the investigation of the motions of the classical rigid body (*) (see footnote on the next page).